A general-covariant concept of particles in curved background

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Abstract

A local current of particle density for scalar fields in curved background is constructed. The current depends on the choice of a two-point function. There is a choice that leads to local non-conservation of the current in a time-dependent gravitational background, which describes local particle production consistent with the usual global description based on the Bogoliubov transformation. Another choice, which might be the most natural one, leads to the local conservation of the current.

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One of the main problems regarding quantum field theory in curved space-time is how to introduce the concept of particles. The problem is related to the fact that particles are *global* objects in the conventional approach [1, 2], while covariance with respect to general coordinate transformations requires local objects. This led some experts to believe that only local operators in field theory were really meaningful and, consequently, that the concept of particles did not have any fundamental meaning [3]. However, particles are what we observe in experiments. Moreover, from the experimental point of view, nothing is more local than the concept of particles. If we require that quantum field theory describes the observed objects, then it should describe particles as local objects. In this letter we show that the concept of particles can be introduced in a local and covariant manner.

Our approach is based on a similarity between the number of particles and charge. For complex fields, the total number of particles is the sum of the number of particles and antiparticles, while the total charge is the difference of these two numbers. The

concept of charge can be described in a local and covariant manner because there exists a local vector current of charge density. We find that a similar vector current exists for the number of particles as well. Nevertheless, it appears that this local current is not unique, but depends on the choice of a two-point function $W(x,x')$. When a unique (or a preferred) vacuum $|0\rangle$ exists, then $W(x,x')$ is equal to the Wightman function $\langle 0|\phi(x)\phi(x')|0\rangle$ and the current is conserved. When such a vacuum does not exist, then there is a choice of W that leads to local non-conservation of the current in a time-dependent gravitational background, which describes local particle production consistent with the usual global description based on the Bogoliubov transformation. Another choice, which might be the most natural one, leads to the local conservation of the current in an arbitrary curved background, provided that other interactions are absent.

Let us start with a hermitian scalar field ϕ in a curved background. The field satisfies the equation:

$$(\nabla^\mu \partial_\mu + m^2 + \xi R)\phi = 0 \;, \tag{1}$$

where R is the curvature. Let Σ be a spacelike Cauchy hypersurface. We define the scalar product

$$(\phi_1, \phi_2) = i \int_\Sigma d\Sigma^\mu \phi_1^* \overleftrightarrow{\partial_\mu} \phi_2 \;, \tag{2}$$

where $\overleftrightarrow{\partial_\mu}$ is the usual antisymmetric derivative. If ϕ_1 and ϕ_2 are solutions of (1), then (2) does not depend on Σ. It is convenient to choose coordinates $(t, \mathbf{x})$ such that $t=$constant on Σ. In these coordinates, the canonical commutation relations can be written as $[\phi(x), \phi(x')]_\Sigma = [\partial_0\phi(x), \partial'_0\phi(x')]_\Sigma = 0$ and

$$\begin{aligned} d\Sigma'^\mu [\phi(x), \partial'_\mu \phi(x')]_\Sigma &= d^3x' \, \tilde{n}^0(x') [\phi(x), \partial'_0 \phi(x')]_\Sigma \\ &= d^3x' \, i\delta^3(\mathbf{x} - \mathbf{x}') \;. \end{aligned} \tag{3}$$

The label Σ denotes that x and x' lie on Σ. The tilde on $\tilde{n}^\mu = |g^{(3)}|^{1/2} n^\mu$ denotes that it is not a vector, while n^μ is a unit vector normal to Σ.

Let us first construct the particle current assuming that a unique (or a preferred) vacuum exists. The field ϕ can be expanded as

$$\phi(x) = \sum_k a_k f_k(x) + a_k^\dagger f_k^*(x) \;, \tag{4}$$

where $(f_k, f_{k'}) = -(f_k^*, f_{k'}^*) = \delta_{kk'}$, $(f_k^*, f_{k'}) = (f_k, f_{k'}^*) = 0$. The operators

$$a_k = (f_k, \phi) \;, \quad a_k^\dagger = -(f_k^*, \phi) \tag{5}$$

satisfy the usual algebra of lowering and raising operators. This allows us to introduce the vacuum as a state with the property $a_k|0\rangle = 0$ and the operator of the total number of particles as

$$N = \sum_k a_k^\dagger a_k \;. \tag{6}$$

We also introduce the function $W(x,x')$, here defined as

$$W(x,x') = \sum_k f_k(x) f_k^*(x') \;. \tag{7}$$

(Later we also study different definitions of $W(x, x')$.) For future reference, we derive some properties of this function. For the fields described by (4), we find

$$W(x, x') = \langle 0|\phi(x)\phi(x')|0\rangle \ . \tag{8}$$

From (7) we find $W^*(x, x') = W(x', x)$. From the fact that f_k and f_k^* satisfy Eq. (1) we find

$$(\nabla^{\mu}\partial_{\mu} + m^2 + \xi R(x))W(x, x') = 0 \ , \tag{9}$$

$$(\nabla'^{\mu}\partial'_{\mu} + m^2 + \xi R(x'))W(x, x') = 0 \ . \tag{10}$$

From (8) and the canonical commutation relations we find that f_k and f_k^* are functions such that

$$\begin{aligned} W(x, x')|_{\Sigma} &= W(x', x)|_{\Sigma}, \\ \partial_0\partial'_0 W(x, x')|_{\Sigma} &= \partial_0\partial'_0 W(x', x)|_{\Sigma}. \end{aligned} \tag{11}$$

$$\tilde{n}^0\partial'_0[W(x, x') - W(x', x)]_{\Sigma} = i\delta^3(\mathbf{x} - \mathbf{x}') \ . \tag{12}$$

So far nothing has been new. However, a new way of looking into the concept of particles emerges when (5) is put into (6) and (2) and (7) are used. This leads to the remarkable result that (6) can be written in the covariant form as

$$N = \int_{\Sigma} d\Sigma^{\mu} j_{\mu}(x) \ , \tag{13}$$

where the vector $j_{\mu}(x)$ is defined as

$$j_{\mu}(x) = \int_{\Sigma} d\Sigma'^{\nu} \frac{1}{2}\{W(x, x')\overleftrightarrow{\partial_{\mu}}\overleftrightarrow{\partial'_{\nu}}\phi(x)\phi(x') + \text{h.c.}\}. \tag{14}$$

Obviously, the hermitian operator $j_{\mu}(x)$ should be interpreted as the local current of particle density. If (4) is used in (14), one can show that $j_{\mu}(x)|0\rangle = 0$.

Assuming that (4) is the usual plane-wave expansion in Minkowski space-time and that the $(t, \mathbf{x})$ coordinates are the usual Lorentz coordinates, we find

$$\tilde{J}_{\mu} \equiv \int d^3x \, j_{\mu}(x) = \sum_{\mathbf{k}} \frac{k_{\mu}}{\omega_{\mathbf{k}}} a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} \ . \tag{15}$$

The quantity $k^i/\omega_{\mathbf{k}}$ is the 3-velocity v^i, so we find the relation $\tilde{\mathbf{J}}|n_q\rangle = \mathbf{v}n_q|n_q\rangle$, where $|n_q\rangle$ is the state with n_q particles with the momentum q. The relations of this paragraph support the interpretation of j_{μ} as the particle current.

Note that although $j_{\mu}(x)$ is a local operator, some non-local features of the particle concept still remain, because (14) involves an integration over Σ on which x lies. Since $\phi(x')$ satisfies (1) and $W(x, x')$ satisfies (10), this integral does not depend on Σ. However, it does depend on the choice of $W(x, x')$. Note also that the separation between x and x' in (14) is spacelike, which softens the non-local features because $W(x, x')$ decreases rapidly with spacelike separation. As can be explicitly seen with the usual plane-wave modes in Minkowski space-time, $W(x, x')$ is negligible when the spacelike separation is much larger than the Compton wavelength m^{-1}.

Using (1) and (9), we find that the current (14) possesses another remarkable property:

$$\nabla^\mu j_\mu(x) = 0 \; . \tag{16}$$

This covariant conservation law means that background gravitational field does not produce particles, provided that a unique (or a preferred) vacuum exists.

To explore the analogy with the current of charge, we generalize the analysis to complex fields, with the expansion

$$\begin{aligned} \phi(x) &= \sum_k a_k f_k(x) + b_k^\dagger f_k^*(x) \; , \\ \phi^\dagger(x) &= \sum_k a_k^\dagger f_k^*(x) + b_k f_k(x) \; . \end{aligned} \tag{17}$$

We introduce two global quantities:

$$N^{(\pm)} = \sum_k a_k^\dagger a_k \pm b_k^\dagger b_k \; . \tag{18}$$

Here $N^{(+)}$ is the total number of particles, while $N^{(-)}$ is the total charge. In a similar way we find the covariant expression

$$N^{(\pm)} = \int_\Sigma d\Sigma^\mu j_\mu^{(\pm)}(x) \; , \tag{19}$$

where

$$\begin{aligned} j_\mu^{(\pm)}(x) \;\; = \;\; & \int_\Sigma d\Sigma'^\nu \frac{1}{2} \{ W(x,x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \\ & [\phi^\dagger(x)\phi(x') \pm \phi(x)\phi^\dagger(x')] + \text{h.c.} \} . \end{aligned} \tag{20}$$

The operator $j_\mu^{(+)}(x)$ is the generalization of (14) with similar properties, including the non-local features. On the other hand, the apparent non-local features of $j_\mu^{(-)}(x)$ really do not exist, because, by using the canonical commutation relations, (11) and (12), $j_\mu^{(-)}(x)$ can be written as

$$\begin{aligned} j_\mu^{(-)}(x) \;\; = \;\; & i\phi^\dagger(x) \overleftrightarrow{\partial_\mu} \phi(x) \\ & + \int_\Sigma d\Sigma'^\nu W(x,x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} W(x',x) \; , \end{aligned} \tag{21}$$

so all non-local features are contained in the second term that does not depend on ϕ. Using (17), (7) and the orthonormality relations among the modes f_k, one can show that the quantity $-\langle 0 | i\phi^\dagger(x) \overleftrightarrow{\partial_\mu} \phi(x) | 0 \rangle$ is equal to the second term, which reveals that the second term represents the subtraction of the infinite vacuum value of the operator represented by the first term. In other words, $j_\mu^{(-)}$ is the usual normal ordered operator of the charge current.

In a way similar to the case of hermitian field, we find the properties $j_\mu^{(+)}|0\rangle = 0$ and $\nabla^\mu j_\mu^{(\pm)} = 0$, while the generalization of (15) is

$$\tilde{J}_\mu^{(\pm)} \equiv \int d^3x \, j_\mu^{(\pm)}(x) = \sum_{\mathbf{k}} \frac{k_\mu}{\omega_{\mathbf{k}}} (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \pm b_{\mathbf{k}}^\dagger b_{\mathbf{k}}) \; . \tag{22}$$

Let us now study the case in which a non-gravitational interaction is also present. In this case, the equation of motion is

$$(\nabla^{\mu}\partial_{\mu} + m^2 + \xi R)\phi = J \,, \tag{23}$$

where $J(x)$ is a local operator containing ϕ and/or other dynamical quantum fields. Since it describes the interaction, it does not contain terms linear in quantum fields. (For example, $J(x)$ may be the self-interaction operator $-\lambda\phi^3(x)$). We propose that even in this general case the currents of particles and of charge are given by the expressions (14) and (20), where $W(x, x')$ is the same function as before, satisfying the "free" equations (9) and (10). As we show below, such an ansatz leads to particle production consistent with the conventional approach to particle production caused by a non-gravitational interaction.

Note also that our ansatz for W makes (21) correct even in the case $J \neq 0$, because the interaction does not modify the canonical commutation relations. This implies that the charge is always conserved, provided that the Lagrangian possesses a global U(1) symmetry. On the other hand, using (9) and (23), we find

$$\nabla^{\mu} j_{\mu}(x) = \int_{\Sigma} d\Sigma'^{\nu} \frac{1}{2} \{ W(x, x') \overleftrightarrow{\partial'_{\nu}} J(x)\phi(x') + \text{h.c.} \} \,, \tag{24}$$

and similarly for $j_{\mu}^{(+)}$. Note that only x (not x') appears as the argument of J on the right-hand side of (24), implying that J plays a strictly local role in particle production.

Let us now show that our covariant description (24) of particle production is consistent with the conventional approach to particle production caused by a non-gravitational interaction. Let $\Sigma(t)$ denote some foliation of space-time into Cauchy spacelike hypersurfaces. The total mean number of particles at the time t in a state $|\psi\rangle$ is

$$\mathcal{N}(t) = \langle \psi | N(t) | \psi \rangle \,, \tag{25}$$

where

$$N(t) = \int_{\Sigma(t)} d\Sigma^{\mu} j_{\mu}(t, \mathbf{x}) \,. \tag{26}$$

Equation (25) is written in the Heisenberg picture. However, matrix elements do not depend on picture. We introduce the interaction picture, where the interaction Hamiltonian is the part of the Hamiltonian that generates the right-hand side of (23). The state $|\psi\rangle$ becomes a time-dependent state $|\psi(t)\rangle$, the time evolution of which is determined by the interaction Hamiltonian. In this picture the field ϕ satisfies the free equation (1), so the expansion (4) can be used. Since we have proposed that W satisfies the free equations (9) and (10), this implies that $N(t)$ becomes $\sum_k a_k^{\dagger} a_k$ in the interaction picture. Therefore, (25) can be written as

$$\mathcal{N}(t) = \sum_k \langle \psi(t) | a_k^{\dagger} a_k | \psi(t) \rangle \,, \tag{27}$$

which is the usual formula that describes particle production caused by a quantum non-gravitational interaction.

Let us now show that (14) can also describe particle production by a classical time-dependent curved background, provided that we take a different choice for the function $W(x,x')$. When the particle production is described by a Bogoliubov transformation, then the preferred modes f_k in (4) do not exist. Instead, one introduces a new set of functions $u_l(x)$ for each time t, such that $u_l(x)$ are positive-frequency modes at that time. This means that the modes u_l possess an extra time dependence, i.e. they become functions of the form $u_l(x;t)$. These functions do not satisfy (1). However, the functions $u_l(x;\tau)$ satisfy (1), provided that τ is kept fixed when the derivative ∂_μ acts on u_l. To describe the local particle production, we take

$$W(x,x') = \sum_l u_l(x;t)u_l^*(x';t') , \tag{28}$$

instead of (7). Since $u_l(x;t)$ do not satisfy (1), the function (28) does not satisfy (9). Instead, we have

$$(\nabla^\mu\partial_\mu + m^2 + \xi R(x))W(x,x') \equiv -K(x,x') \neq 0 . \tag{29}$$

Using (1) and (29) in (14), we find a relation similar to (24):

$$\nabla^\mu j_\mu(x) = \int_\Sigma d\Sigma'^\nu \frac{1}{2}\{K(x,x')\overleftrightarrow{\partial'_\nu}\phi(x)\phi(x') + \text{h.c.}\} . \tag{30}$$

This local description of the particle production is consistent with the usual global description based on the Bogoliubov transformation. This is because (26) and (14) with (28) and (4) lead to

$$N(t) = \sum_l A_l^\dagger(t)A_l(t) , \tag{31}$$

where

$$A_l(t) = \sum_k \alpha_{lk}^*(t)a_k - \beta_{lk}^*(t)a_k^\dagger , \tag{32}$$

$$\alpha_{lk}(t) = (f_k,u_l) , \quad \beta_{lk}(t) = -(f_k^*,u_l) . \tag{33}$$

The time dependence of the Bogoliubov coefficients $\alpha_{lk}(t)$ and $\beta_{lk}(t)$ is related to the extra time dependence of the modes $u_l(x;t)$. If we assume that the change of the average number of particles is slow, i.e. that $\partial_t A_l(t) \approx 0$, $\partial_t u_l(\tau,\mathbf{x};t)|_{\tau=t} \approx 0$, then the Bogoliubov coefficients (33) are equal to the usual Bogoliubov coefficients. This approximation is nothing else but the adiabatic approximation, which is a usual part of the convential description of particle production [4].

In general, there is no universal natural choice for the modes $u_l(x;t)$. In particular, in a given space-time, the choice of the natural modes $u_l(x;t)$ may depend on the observer. If different observers (that use different coordinates) use different modes for the choice of (28), then the coordinate transformation alone does not describe how the particle current is seen by different observers. In this sense, the particle current is not really covariant. There are also other problems related to the case in which different observers use different modes [5].

Since the covariance was our original aim, it is desirable to find a universal natural choice of $W(x,x')$, such that, in Minkowski space-time, it reduces to the usual plane-wave expansion (7). Such a choice exists. This is the function $G^+(x,x')$ satisfying

(9) and (10), calculated in a well-known way from the Feynman propagator $G_F(x,x')$ [2, 6]. The Feynman propagator $G_F(x,x')$, calculated using the Schwinger-DeWitt method, is unique, provided that a geodesic connecting x and x' is chosen. When x and x' are sufficiently close to one another, then there is only one such geodesic. In this case, the adiabatic expansion [2, 6] of the Feynman propagator can be used. For practical calculations, the adiabatic expansion may be sufficient because $G^+(x,x')$ decreases rapidly with spacelike separation, so the contributions from large spacelike separations may be negligible.

To define the exact unique particle current, we need a natural generalization of $G_F(x,x')$ to the case with more than one geodesic connecting x and x'. The most natural choice is the two-point function $\tilde{G}_F(x,x')$ defined as the average over all geodesics connecting x and x'. Assuming that there are N such geodesics, this average is

$$\tilde{G}_F(x,x') = N^{-1} \sum_{a=1}^{N} G_F(x,x';\sigma_a) , \tag{34}$$

where $G_F(x,x';\sigma_a)$ is the Feynman propagator $G_F(x,x')$ calculated with respect to the geodesic σ_a. Eq. (34) can be generalized even to the case with a continuous set of geodesics connecting x and x'. This involves a technical subtlety related to the definition of measure on the set of all geodesics, which we shall explain in detail elsewhere.

Since the function $\tilde{G}^+(x,x')$ calculated from $\tilde{G}_F(x,x')$ always satisfies (9) and (10), it follows that the corresponding particle current is conserved and does not depend on the choice of Σ, provided that the field satisfies (1). Note that this definition of particles does not always correspond to the quantities detected by "particle detectors" of the Unruh-DeWitt type [7, 8]. Instead, this number of particles is determined by a well-defined hermitian operator that does not require a model of a particle detector, just as is the case for all other observables in quantum mechanics. Moreover, since (14) does not require a choice of representation of field algebra, the definition of particles based on $\tilde{G}_F$ does not require the choice of representation either. This allows us to treat the particles in the framework of algebraic quantum field theory in curved background [9]. Furthermore, as noted by Unruh [7], only one definition of particles can correspond to the real world, in the sense that their stress-energy contributes to the gravitational field. Since the definition of particles based on $\tilde{G}_F$ is universal, unique and really covariant, it might be that these are the particles that correspond to the real world. Besides, as we shall discuss elsewhere, it seems that these particles might correspond to the objects detected by real detectors (such as a Wilson chamber or a Geiger-Müller counter) in real experiments.

In this letter we have constructed the operator describing the local current of particle density, which allows us to treat the concept of particles in quantum field theory in a local and generally covariant manner. This operator is not unique, but depends on the choice of the two-point function $W(x,x')$. Different choices correspond to different definitions of particles. In particular, various choices based on (28) correspond to particle production by the gravitational field. This local description of particle production is consistent with the conventional global description based on

the Bogoliubov transformation. Similarly, various choices based on (7) give a local description of particle content in various inequivalent representations of field algebra.

We have seen that a particularly interesting choice of $W(x,x')$ is that based on the Feynman propagator $\tilde{G}_F(x,x')$, because this choice might correspond to the most natural universal definition of particles for quantum field theory in curved background. It is tempting to interpret these particles as real particles. If this interpretation is correct, then classical gravitational backgrounds do not produce real particles. There are also other indications that classical gravitational backgrounds might not produce particles [10, 11, 12]. Even if the particles based on $\tilde{G}_F$ do not correspond to real physical particles in general, it is interesting to ask about the physical meaning of this time-independent hermitian observable that corresponds to physical particles at least in Minkowski space-time.

Note finally that our definition of the particle current can be generalized to the case of scalar and spinor fields in classical electromagnetic backgrounds. Again, a two-point function can be chosen such that the particle production is described in a way consistent with the Bogoliubov-transformation method, but the requirement of gauge invariance leads to a conserved current, indicating that classical electromagnetic backgrounds also might not produce particles. Only the quantized electromagnetic field (having a role similar to $J(x)$ in (23)) may cause particle production in this formalism, which is in agreement with some other results [12]. This will be discussed in more detail elsewhere.

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